MODE SHAPE OF DUCT ACOUSTICS

WHY STUDY THE ACOUSTICS OF DUCTS

Ducts, also known as waveguides, are able to efficiently transmit sound over large distances. Some common examples:

- Ventilation ducts
- Exhaust ducts
- Automotive silencers
- Shallow water channels and surface ducts in deep water
- Turbofan engine ducts

WAVE EQUATION

The acoustic pressure $p(x,y,z,t)$ in a source-free region of space in which there is a uniform mean flow U_x in the x – direction satisfies the convected wave equation:

$$\left[\frac{D^2}{Dt^2} - c^2\nabla^2\right]p = 0 \qquad \frac{Dp}{Dt} = \frac{\partial p}{\partial t} + U_x\frac{\partial p}{\partial x}$$

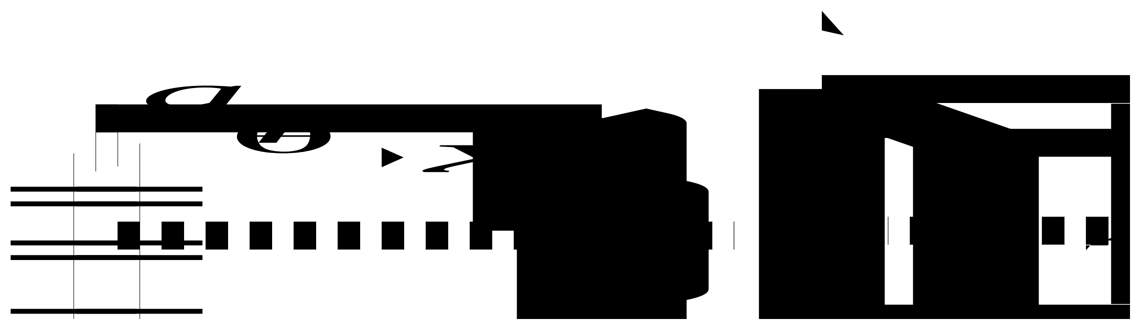
where c is the speed of sound.

For simplicity, and without loss of generality, we shall only consider solutions to the wave equation in the absence of flow, $U_x = 0$.

SEPARABLE SOLUTIONS TO THE WAVE EQUATION IN CARTESIAN AND CYLINDRICAL COORDINATES

Cylindrical duct

Rectangular duct



$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Assume harmonic (single-frequency) separable solutions of the form

$$p = p_0 e^{i(\alpha k x - \omega t)} G(r) H(\theta)$$

$$p = p_0 e^{i(\alpha k x - \omega t)} Y(y) Z(z)$$

SEPARABLE SOLUTIONS TO THE WAVE EQUATION IN CARTESIAN AND CYLINDRICAL COORDINATES

(Continued)

Substituting into the wave equation and separating variables:

Cylindrical duct

$$\frac{\partial^2 G}{\partial^2 r} + \frac{1}{r}\frac{\partial G}{\partial r} + \left\{k^2\left(1-\alpha^2\right) - \frac{m^2}{r^2}\right\}G = 0$$

Bessel's equation of order *m*

$$\frac{\partial^2 H}{\partial^2 \theta} + m^2 H = 0$$

Rectangular duct

$$\frac{\partial^2 Y}{\partial^2 y} + k_x^2 Y = 0$$

$$\frac{\partial^2 Z}{\partial^2 z} + k_z^2 Z = 0$$

GENERAL SOLUTIONS

General solutions to these second order equations are:

$$\left.\begin{aligned} G(r) &= \begin{matrix} J_m(k_r r) \\ Y_m(k_r r) \end{matrix} \\ \\ H(\theta) &= e^{im\theta} \end{aligned}\right\} k_r^2 = k^2\left(1-\alpha^2\right)$$

$$\left.\begin{aligned} Y(y) &= \begin{matrix} \sin(k_y y) \\ \cos(k_y y) \end{matrix} \\ \\ Z(z) &= \begin{matrix} \sin(k_z z) \\ \cos(k_z z) \end{matrix} \end{aligned}\right\} k_y^2 + k_z^2 = k^2\left(1-\alpha^2\right)$$

Transverse wavenumber is k_r.

Transverse wavenumber is $\sqrt{k_y^2 + k_z^2}$

HARD-WALLED DUCT EIGENVALUE EQUATION

The component of particle velocity u_n normal to the hard-walled duct vanishes:

$$\frac{\partial G}{\partial r} = 0 \qquad \frac{\partial Z}{\partial z} = 0 \qquad \frac{\partial Y}{\partial y} = 0$$

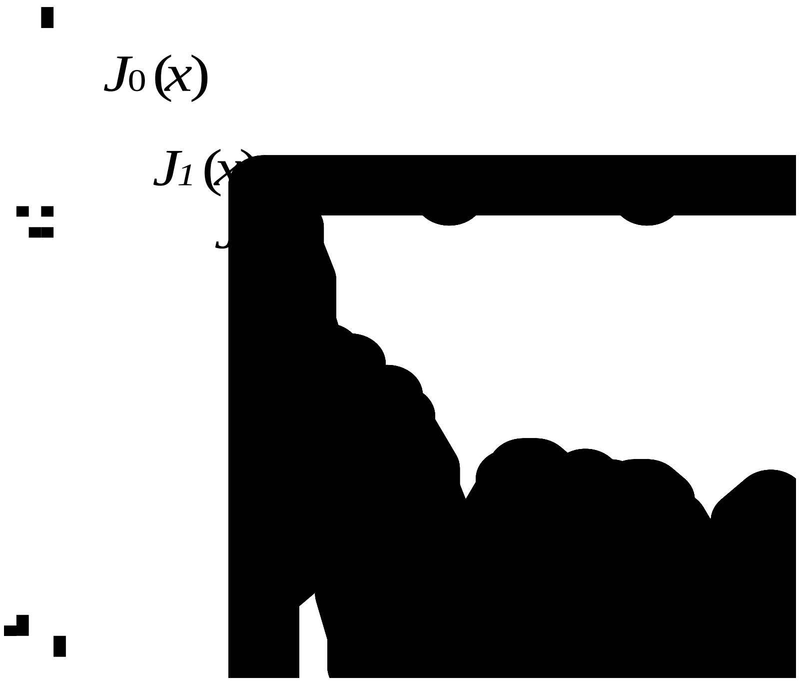
The solutions $Y_m(k_r r)$, $\sin(k_z z)$ and $\sin(k_y z)$ cannot satisfy these boundary conditions. Furthermore, only particular discrete values of transverse wavenumbers (eigenvalues) satisfy the boundary conditions given by.

$$J'_m\left(k_{mn}a\right) = 0 \qquad \sin\left(k_{ny}L_y\right) = 0 \qquad \sin\left(k_{nz}L_z\right) = 0$$

MODAL EIGENVALUES

$$k_{rmn} = j'_{mn} / a \qquad k_{ny} = n_y \pi / L_y \qquad k_{nz} = n_z \pi / L_z$$

Bessel functions of order m = 0, 1 and 2



Stationary values of the Bessel function

m/n	**1**	**2**	**3**	**4**	**5**	**6**	**7**
0	0	3.8317	7.0156	10.1735	13.3237	16.4706	19.6159
1	1.8412	5.3314	8.5363	11.7060	14.8636	18.0155	21.1644
2	3.0542	6.7061	9.9695	13.1704	16.3475	19.5129	22.6716
3	4.2012	8.0152	11.345	14.5858	17.7887	20.9725	24.1449
4	5.3176	9.2824	12.681	15.9641	19.1960	22.4010	25.5898
5	6.4156	10.5199	13.987	17.3128	20.5755	23.8036	27.0103
6	7.5013	11.734	15.268	18.6374	21.9317	25.1839	28.4098

CUT OFF FREQUENCY

Earlier we saw that the transverse and axial wavenumbers of a single mode are connected by the dispersion relationships

Cylindrical duct

$$\alpha_{mn} = \sqrt{1-\left(j'_{mn}/ka\right)^2}$$

Rectangular duct

$$\alpha_{nynz} = \sqrt{1-\left[\left(n_y\pi/kL_y\right)^2+\left(n_z\pi/kL_z\right)^2\right]}$$

These expressions make explicit the existence of threshold frequencies $\omega_{mn} = ck_{mn}$ at frequencies *below* which α is purely imaginary and the mode decays exponentially along the duct. The mode is said to cut off, or evanescent.

CUT OFF FREQUENCY

This cut-off frequency follows from the above as

Cylindrical duct

$$\omega_{mn} = c j'_{mn} / a$$

Rectangular duct

$$\omega_{nynz} = c\pi \left[\left(n_y / L_y \right)^2 + \left(n_z / L_z \right)^2 \right]^{-1/2}$$

In terms of cut-off frequency

$$\alpha_{mn} = \sqrt{1 - \left(\omega_{mn} / \omega \right)^2}$$

$$\alpha_{nynz} = \sqrt{1 - \left(\omega_{nynz} / \omega \right)^2}$$

MODE COUNT FORMULAE

At a single frequency only a finite number of modes $N(ka)$ and $N(kL_x)$ are cut on and able to propagate along the duct without attenuation. The rest decay exponentially along the duct. In the high frequency limit:

Cylindrical duct

$$N(ka) = \tfrac{1}{2}ka + \left(\tfrac{1}{2}ka\right)^2$$

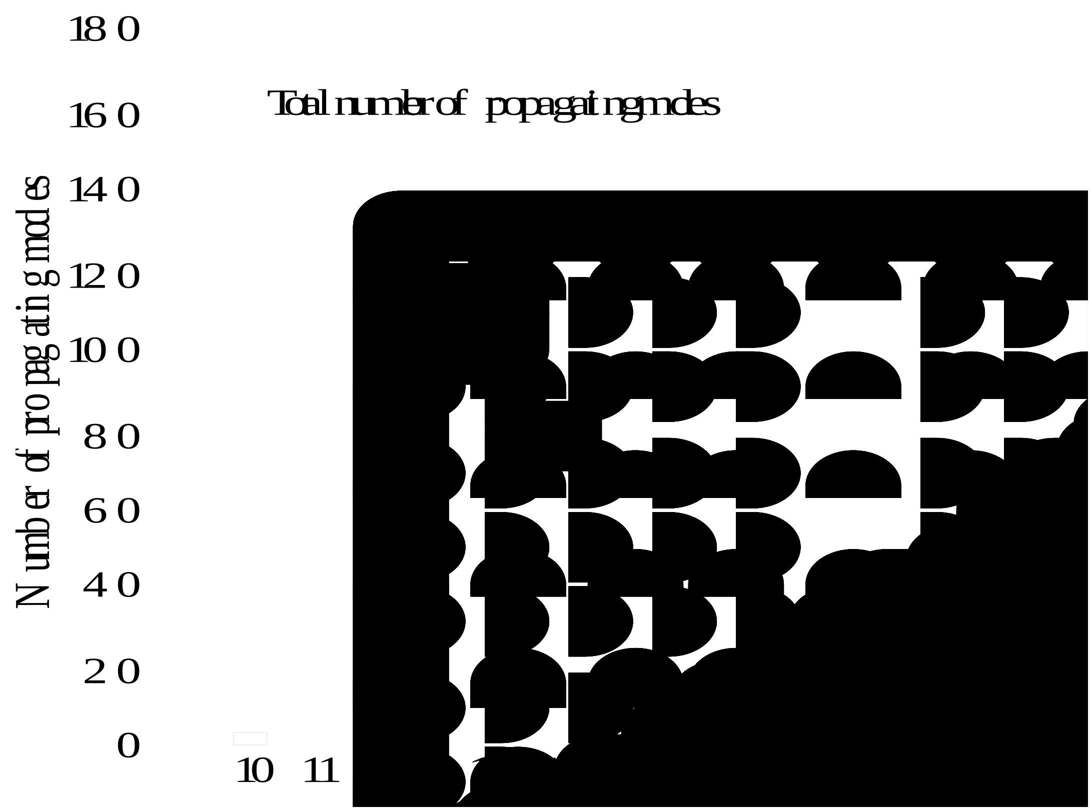
Rectangular duct

$$N(kL_x) = \frac{(R+1)}{\pi}kL_x + \frac{R}{\pi}(kL_x)^2$$

where $R = L_y / L_x \geq 1$

MODE COUNT FORMULAE

A comparison of this mode-count formula for circular ducts with the exact count (histogram) is presented below.



MODES AND MODE SHAPE FUNCTIONS

In seeking a solution for the pressure field in a duct we obtained, not a single unique solution, but a family of solutions. The general solution is a linear superposition of these 'eigenfunction' solutions:

Cylindrical duct

$$p(r,\theta,x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \overline{p}_{mn} \Phi_{mn}(r,\theta) e^{-i\alpha_{mn} kx}$$

$$\Phi_{mn}(k_{rmn} r) = J_m(k_{rmn} r) e^{im\theta}$$

Rectangular duct

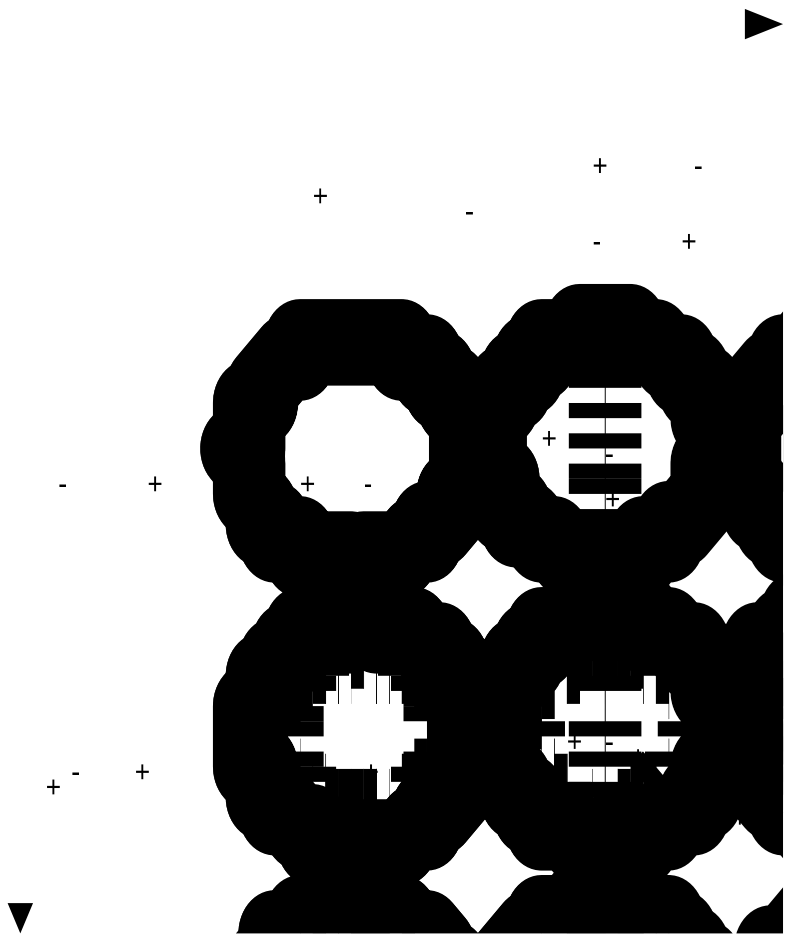
$$p(y,z,x) = \sum_{ny=1}^{\infty} \sum_{nz=1}^{\infty} \overline{p}_{nxny} \Phi_{nxny}(y,z) e^{-i\alpha kx}$$

$$\Phi_{nynz}(y,z) = \cos(k_{ny} y) \cos(k_{nz} z)$$

The resultant acoustic pressure in the duct is the weighted sum of fixed pressure patterns across the duct cross section. Each of which propagate axially along the duct at their characteristic axial phase speeds.

MODE SHAPE FUNCTIONS Φ

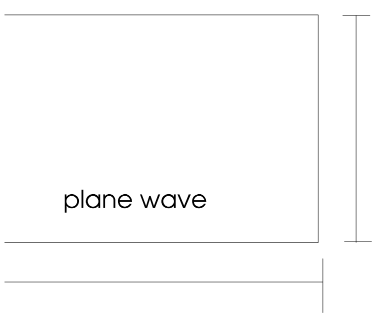
Cylindrical Duct Mode Shape Functions

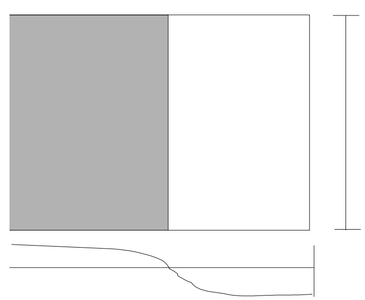


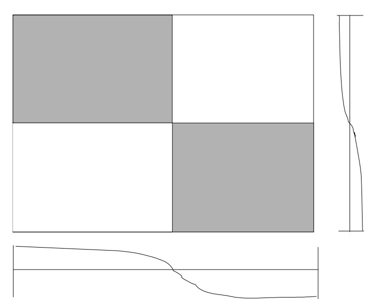
MODE SHAPE FUNCTIONS Φ

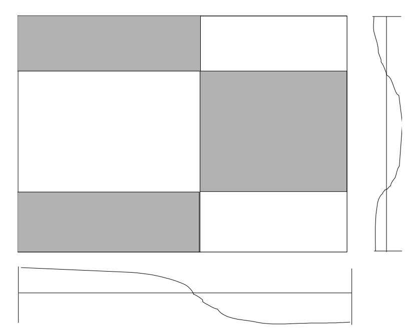
Rectangular Duct Mode Shape Functions

$n_y = 0, n_z = 0$ $n_y = 0, n_z = 1$ $n_y = 1, n_z = 1$ $n_y = 1, n_z = 2$

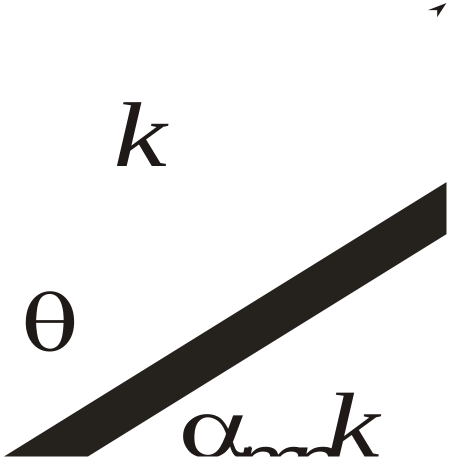








CUT-OFF RATIO



Earlier we saw that each mode the axial wavenumber is $\alpha_{mn}k$ and the wavenumber in the direction of propagation is k (=ω/c)

By simple geometry, α_{mn} equals the cosine of the angle θ between the local modal wavefront and the duct axis. It may therefore be interpreted as a measure of how much the mode is cut on.

A more common index of cut on is specified by the cut-off ratio ζ defined by $\zeta_{mn} = 1/\sqrt{1-\alpha_{mn}^2}$

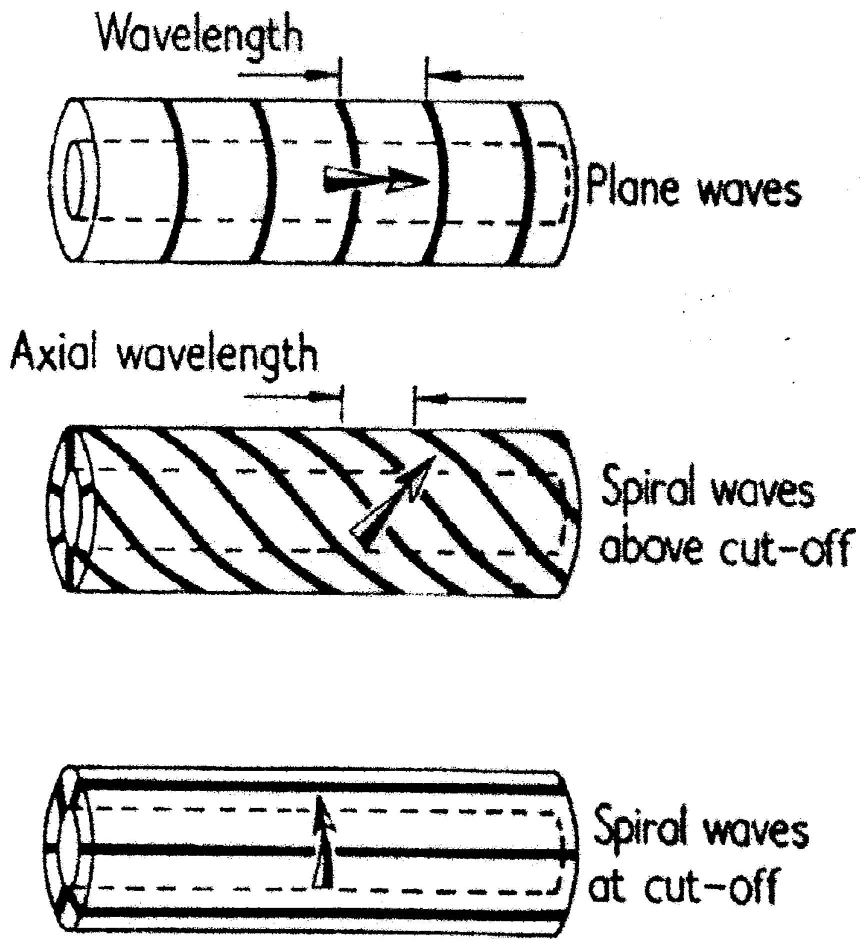
CATEGORIES OF MODAL BEHAVIOUR

$\zeta_{mn} < 1.$ Mode is cut-off and decays exponentially along the duct. Pressure and particle velocity are in quadrature and zero power is transmitted.

$\zeta_{mn} = 1.$ Mode is just cut on (or cut-off) and propagates with infinite phase speed (and zero group velocity). No modal power is transmitted.

$\zeta_{mn} > 1.$ Mode is cut on and propagates at an angle $\cos^{-1}\alpha_{mn}$ to the duct axis. Transmitted modal sound power . Axial phase speed greater than c and group velocity less than c.

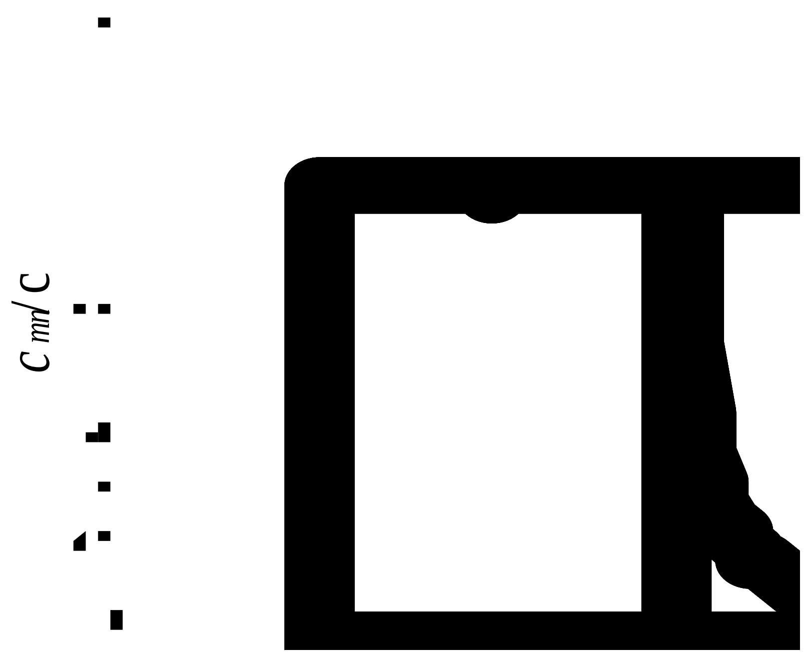
CATEGORIES OF MODAL BEHAVIOUR



AXIAL PHASE SPEED

Each mode propagates axially along the duct as $p_{mn} \propto \mathrm{e}^{-i\alpha_{mn}kx}$.
The axial modal phase speed c_{mn} is given by

$$c_{mn} = \frac{\omega}{\alpha_{mn}k} = \alpha_{mn}^{-1}c \qquad \frac{c_{mn}}{c} = \frac{1}{\sqrt{1-\left(\omega_{mn}/\omega\right)^2}}$$



The axial phase is infinite at the cut-off frequency tending to c as the frequency approaches infinity.

CIRCUMFERENTIAL PHASE SPEED

At a fixed position along the duct, points at the wall of constant phase are given by $\omega t - m\phi$. The circumferential phase speed at the wall is therefore

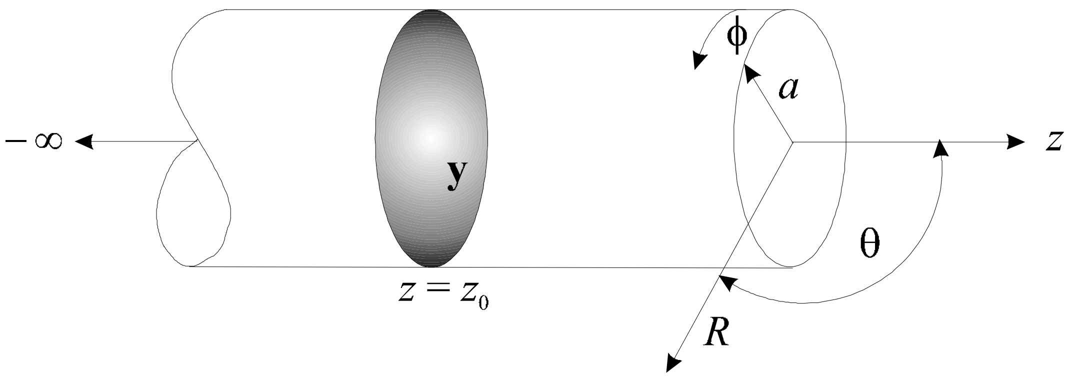
$$V_p = a\frac{\partial\phi}{\partial t} = \frac{\omega a}{m}$$

A property of Bessel functions J_m for large m is that $k_{m1}a \geq m$.

Combining this result with the cut on condition $\omega / c > k_{m1}$ gives

$$\frac{\omega a}{mc} > 1 \qquad \text{and hence} \qquad V_p > c$$

MODAL RADIATION FROM HARD WALLED CIRCULAR DUCTS

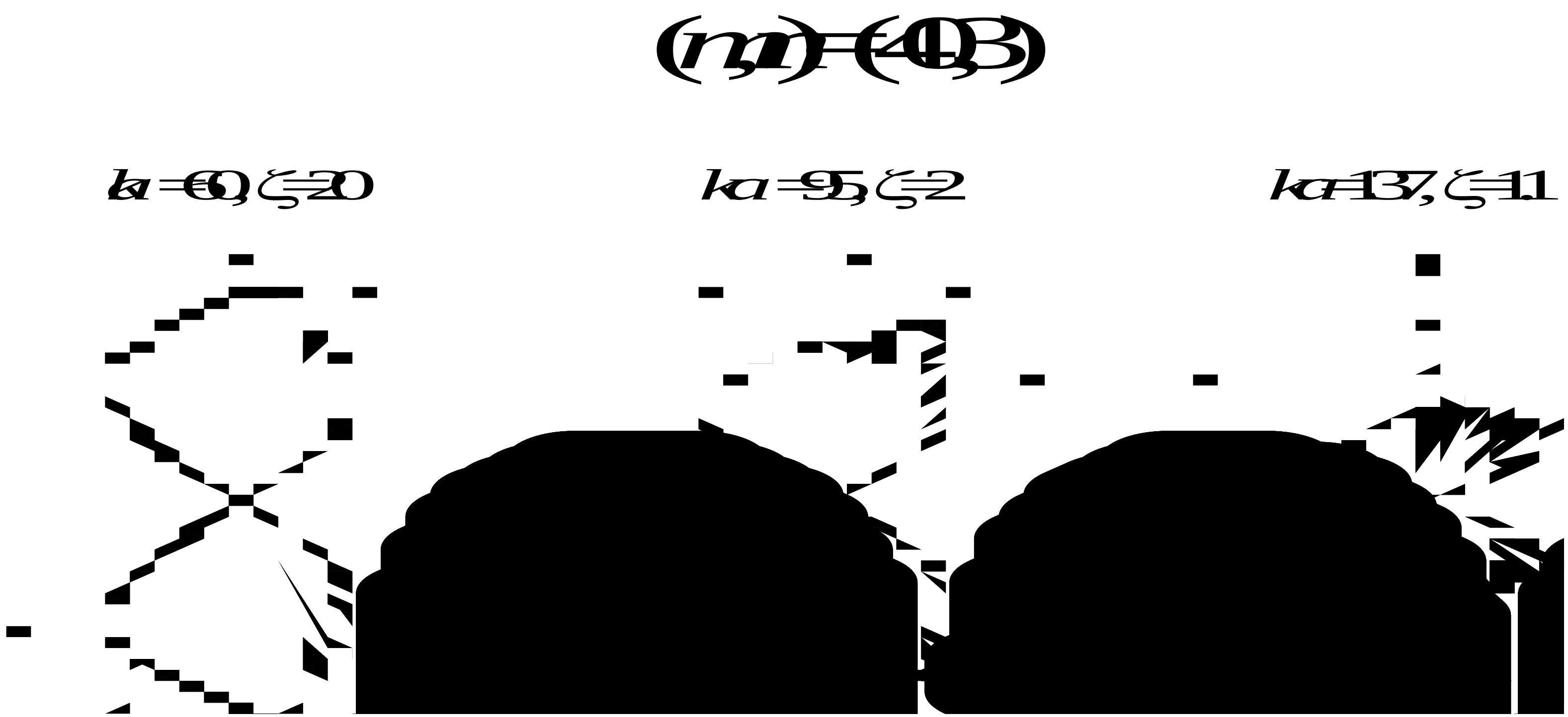


The acoustic pressure $p(R,\theta,\phi)$ in the far field of a semi-infinite circular hard walled duct may be expressed as

$$p_{mn}(R,\theta,\phi) = p_{mn} D_{mn}(k,a,\phi) \frac{e^{im\theta - ikR - i\omega t}}{R} ; kR >> 1$$

MODAL RADIATION FROM HARD WALLED CIRCULAR DUCTS

Some representative directivity plots (in decibels) for the $(m,n)=(40,3)$ mode at three frequencies is presented below



MODAL RADIATION FROM HARD WALLED CIRCULAR DUCTS
SIMPLE RULES

- The angle θ_P of the principal radiation lobe equals $\theta_P = \sin^{-1}\zeta_{mn}^{-1}$ which is identical to the axial propagation angle within the duct.

- Modal radiation becomes progressively weaker as the frequency approach cut-off from above tending to zero exactly at cut-off.

- No major or minor lobes occur in the rear arc.

- Zeros (or nulls) in the radiation pattern occur at angles$= \cos^{-1} j'_{mj}, \quad j \neq n$. Angles of the minor lobes occur roughly mid-way between the angles of the zeros. The number of zeros and minor lobes increase roughly as the frequency squared.

- Symmetrical angles exist θ_s beyond which modal radiation is extremely weak. These are referred to as shadow zones (or cones of silence) and occur at $\theta_s = \sin^{-1} m/ka$